Singularities in Multifractal Turbulence—Dissipation Networks and Their Degeneration

*A. Bershadskii** and *C. H. Gibson***

P.O.Box 39953, *Ramat* – *Aviv* 61398,

Tel – *Aviv*, *Israel**

Departments of Applied Mechanics and Engineering Sciences

and Scripps Institution of Oceanography,

University of California at San Diego, 0411, *La Jolla*,

California 92093 – 0411, *U.S.A.***

Abstract

We suggest that large-scale turbulence dissipation is concentrated along caustic networks (that appear due to vortex sheet instability in three-dimensional space), leading to an effective fractal dimension $D_{eff} = 5/3$ of the network backbone and a turbulence intermittency exponent $\mu = 1/6$. Actually, $D_{eff} < 5/3$ and $\mu > 1/6$ due to singularities on these caustic networks. It is shown (using the theory of caustic singularities) that the strongest (however, stable on the backbone) singularities lead to $D_{eff} = 4/3$ (an elastic backbone) and to $\mu = 1/3$. Thus, there is a restriction of the network fractal variability: $4/3 < D_{eff} < 5/3$, and consequently: $1/6 < \mu < 1/3$.

Degeneration of these networks into a system of smooth vortex filaments: $D_{eff} = 1$, leads to $\mu = 1/2$. After degeneration, the strongest singularities of the dissipation field, ε, lose their power-law form, while the smoother field $ln\varepsilon$ takes it. It is shown (using the method of multifractal asymptotics) that the probability distribution of the dissipation changes its form from exponential-like to log-normal-like with this degeneration, and that the multifractal asymptote of the field $ln\varepsilon$ is related to the multifractal asymptote of the energy field.

Finally, a phenomenon of acceleration of large-scale turbulent diffusion of passive scalar by the singularities is briefly discussed.

All results are based on experimental data.

1 Introduction

The turbulence intermittency exponent μ can be defined from the spectral density of the dissipation field ε [1]

$$E_{\varepsilon} \sim k^{-1} \ (Lk)^{\mu} \tag{1}$$

where k is the wave number and L is the largest space scale of the turbulence. This exponent is rather well measured and could be a useful characteristic of turbulent intermittency (if one could relate it to the geometrical characteristics of the dissipation field). Numerous turbulence experiments give two 'attracting' values of the index: $\mu \simeq 0.25 \pm 0.10$ and $\mu \simeq 0.45 \pm 0.05$ (see, for example, [1] - [4]). Comparative analysis of different experimental data (performed in [5]) indicates that the first 'attractive' value of μ characterizes relatively large space scales while the second characterizes relatively small scales. It should be noted, however, that both scale intervals satisfy the condition $Lk \gg 1$ (which is significant for the following considerations). One might expect that these two ranges of scales reflect differences in the geometrical (fractal) structure of the dissipation field. In fact, we show in the paper that this difference can be related to different classes of singularities in the dissipation field. Moreover, the above mentioned variability of μ for the large-scale range may also be related to different types of singularities within a class.

To obtain relations between the intermittency exponent μ and the fractal (singular) characteristics of the dissipation field let us recall that turbulent dissipation is concentrated on manifolds with dimensions that are much less than three (in three-dimensional space): vortex sheets and vortex filaments. However, vortex sheets are unstable in three-dimensional space. The instabilities have a large-scale wave nature (see for example [6]). The stochastic waves themselves have an intermittent structure and it is natural to suggest that the main part of the dissipation is concentrated along the boundaries between subregions with a complicated wave structure (due to an interference phenomenon), and subregions without these waves. Such boundaries are called caustics.

When caustic networks are formed various geometrical singularities appear in the dissipation field. These singularities strongly affect dissipation spectra and the effective fractal dimension of the networks. If a passive scalar field like temperature is also concentrated along the caustics a phenomenon of accelerated turbulent diffusion by the singularities takes place. Degeneration of these networks leads to dramatic changes in all of these processes.

2 Eikonal representation

The spectral density E_ε is defined as

$$E_\varepsilon = \frac{1}{8\pi^3} \int_{|\mathbf{k}|=k} d\mathbf{k} \int d\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}}\ R_\varepsilon(\mathbf{r}) \tag{2}$$

where the space autocorrelation of energy dissipation is

$$R_\varepsilon(\mathbf{r}) = \langle \varepsilon(\mathbf{x})\varepsilon(\mathbf{x}+\mathbf{r})\rangle. \tag{3}$$

For $k \to \infty$ the multiplier $\exp(-i\mathbf{k}\cdot\mathbf{r})$ in (2) is a rapidly oscillating function, and the principal contribution to the asymptotic behavior of $E_\varepsilon(k)$ is given by $R_\varepsilon(\mathbf{r})$ with small r. Let us recall that we consider the case $Lk \gg 1$ even when we speak about relatively large scales. The idea of an eikonal approximation (see, for example, [7]) is to study the critical points of a corresponding eikonal function $\phi(\mathbf{r})$ instead of the behavior of the autocorrelation $R_\varepsilon(\mathbf{r})$ at small r. The eikonal representation for $k \to \infty$ is

$$E_\varepsilon(k) \propto \int_{|\mathbf{k}|=k} d\mathbf{k} \int d\mathbf{r}\ e^{-i\mathbf{k}\cdot\mathbf{r}}\ R_\varepsilon\ (\mathbf{r}) \to \int d\mathbf{r}\ f(\mathbf{r}) e^{ik\phi(\mathbf{r})} \tag{4}$$

where $f(\mathbf{r})$ is an infinitely differentiable function of compact support and $\phi(\mathbf{r})$ is a smooth function. By a stationary phase principal, the main contribution to the asymptotic behavior of the integral as $k \to \infty$ is given by neighborhoods of the critical points of ϕ [7]. Recall that a critical point of a smooth function is a point at which the differential of the function is equal to zero. A critical point is non-degenerate (generic) if the second differential is a non-degenerate quadratic form. Functions in general position have only non-degenerate critical points [7]. If the function $\phi(\mathbf{r})$ is a function of general position then integral (4) has a simple (power-law) asymptote with $k \to \infty$ [7]:

$$E_\varepsilon(k) \propto k^{-D/2} \tag{5}$$

where D is the dimension of the space support in the vicinity of the critical point. It should be noted that every degenerate critical point splits into several non-degenerate critical points under a small perturbation (the famous Morse theorem) [7]. Generally,

D is the fractal dimension, and has different values in neighborhoods of different critical points. Let us assume that D is the same in small neighborhoods of all critical points. This assumption is consistent with a multifractal picture if all critical points belong to some specific manifold. One can see from (5) that for vortex sheets, $D = 2$, we obtain from (5) spectrum (1) with $\mu = 0$, while for smooth vortex filaments, $D = 1$, it follows from (1) and (5) that $\mu = 0.5$.

3 Singularities on caustics

If the critical points belong to the caustics, but are generic nonsingular ones, than the integral (4) has an asymptotic estimate in the form [7]

$$E_\varepsilon(k) \propto k^{-D/2+1/6}. \tag{6}$$

Thus, for a quasi two-dimensional situation (arising due to the large-scale wave instability of vortex sheets)

$$E_\varepsilon(k) \propto k^{-1+1/6}, \tag{7}$$

i.e., $\mu = 1/6$ (cf. (1)). One can obtain an effective fractal dimension of caustic networks by comparison (5) and (7): $D_{eff}/2 = 5/6$ so $D_{eff} = 5/3$. This fractal dimension is close to the fractal dimension of so-called backbones of percolation networks [8] and there are fundamental reasons for such closeness (see, for example, [9]-[12]).

The existence of singularities on caustics can lead to an apparent variability of μ and D_{eff}. Moreover, in this case $\mu > 1/6$ [7] and, consequently, $D_{eff} < 5/3$. There are only four types of *staaa* caustic singularities in three-dimensional space (every more complicated type of caustic singularity will break down into these *saa aast* ones if the generic position is disturbed slightly). These singularities are called: cusp ridge, swallowtails, pyramid and purse points [7]. Values of $\mu = 1/4$ for ordinary points along a caustic cusp ridge; $\mu = 0.3$ for swallowtails; and for pyramid and purse points, $\mu = 1/3$ [7]. The exponent μ measures the degree to which dissipation is concentrated toward the caustic. Thus, one can see that the minimal effective fractal dimension $D_{eff} = 4/3$ is given by pyramid and purse singularities. At a pyramid singularity, three (smooth) cusp ridges are mutually tangent while in the neighborhood of a purse point, the caustic comprises two surfaces with similar singularities, intersecting along a pair of curves (the cusp ridges of the two surfaces continue each other, forming a single smooth curve). Three-dimensional topological possibilities are necessary for the existence of these singularities. Corresponding *a aaaa aa* effective fractal dimension $D_{eff} = 4/3$ is close to the fractal dimension of the so-called elastic backbone of percolation networks in three-dimensional space [13] (cf. also [9],[11]). This coincidence is also quite clear because the elastic backbone of a cluster can be defined as being the sites that lie on the union of all the *saartast* paths between two points on the cluster [13] (see also Discussion).

In this way we obtain an upper bound for the caustic intermittency index $1/6 < \mu < 1/3$ which corresponds to the minimal $D_{eff} = 4/3$ (defined by the pyramid and purse caustic singularities). This range of μ-variability coincides with the corresponding large-scale range of experimentally observed values of μ mentioned in the Introduction.

4 Caustic network degeneration, multifractality and probability distributions

New (sharper) *staaaa* singularities can appear only with degeneration of the caustic networks. A limiting situation that can occur with this degeneration is a system of smooth vortex filaments with effective fractal dimension $D_{eff} = 1$ and $\mu = 1/2$ (see Section 2). In [17] a relation between the vortex-tube filamentation process and multifractality of dissipation fields has been considered and the existence of a multifractal asymptote (i.e., D_{∞}, see below) of the dissipation field has been related to this filamentation process. If, however, the vortex filaments occur due to caustic network degeneration (i.e., without the filamentation process) then the corresponding singularity of the dissipation field can be so strong that there is no definite value of D_{∞}. In this case the smoother field $ln\varepsilon$ can assume multifractal properties with a definite value of the D_{∞}. In this section we will study such a situation.

In the paper [14] self-similar behavior of subregions with large values of energy, energy dissipation and vorticity (enstrophy) were established based on experimental turbulence data. Namely, let us divide a region occupied by a turbulent flow into a set of subvolumes V_i with characteristic scale r and define a local mean of some (positive) scalar field $g(\mathbf{x})$ over each volume

$$g_i(r) = r^{-d} \int_{V_i} g(\mathbf{x}) d\mathbf{x} \tag{8}$$

and a q-moment as

$$\langle g_r^q \rangle = (L/r)^{-d} \sum_i g_i(r)^q \tag{9}$$

where d is the dimension of space and L is a characteristic scale of the region. The multifractal hypothesis for the field g is the suggestion that for $(r/L) \to 0$ [15]

$$\langle g_r^q \rangle \sim (r/L)^{-\mu_q} \tag{10}$$

where

$$\mu_q = (d - D_q)(q - 1) \qquad (11).$$

(D_q is the so-called generalized dimension). For $q > 0$ the value of $D_q \leq d$. It has been shown in [14] (for moderate values of Reynolds number) that the subregions with large values of g_i (where g is the turbulent energy, energy dissipation or enstrophy) have self-similar behavior and, therefore, one can define the multifractal asymptote

$$large \ g_i(r) \sim r^{D_\infty - d}. \tag{12}$$

This asymptote corresponds to the strongest power-law singularity in the field $g(\mathbf{x})$ since $D_\infty \leq D_q$ for all q.

In the following we will deal only with *large* $g_i(r)$. Therefore, for convenience, we will not write *large* in the corresponding places. It is well established that the probability density function (PDF) for turbulent energy has an exponential form for $L/r \gg 1$ (at least for large enough values of energy; see, for example, [14] and references therein). This allows us to use the exponential asymptote of the energy PDF as a standard

$$P(g^{(e)}) \sim e^{-ag^{(e)}} \tag{13}$$

where $g^{(e)} = \mathbf{u}^2$. The multifractality of turbulent energy fluctuations was first studied in the paper [16]. In [14] the existence of a multifractal asymptote for turbulence energy fluctuations was shown; that is, the singular (power-law) estimate

$$g_r^{(e)} \sim r^{(D_\infty^{(e)} - d)} \tag{14}$$

was found for large enough values of $g_r^{(e)}$. An analogous estimate was also established in [14] for large fluctuations of turbulence energy dissipation ε (at least for moderate values of Reynolds number, see also [17])

$$\varepsilon_r \sim r^{(D_\infty^{(\varepsilon)} - d)}. \tag{15}$$

The simplest approach is to use asymptotic representations (14) and (15) to obtain an asymptotic PDF of the ε-field from the standard asymptotic PDF for large energy fluctuations (13). In this case, the space scale r plays an intermediary role. Thus one can obtain

$$P(\varepsilon) \sim exp\{-b\varepsilon^{\gamma}\} \tag{16}$$

where

$$\gamma = \frac{D_\infty^{(e)} - d}{D_\infty^{(\varepsilon)} - d}. \tag{17}$$

If we take experimental values for $D_\infty^{(\varepsilon)} - d \simeq -2/3$ and for $D_\infty^{(e)} - d \simeq -1/3$ ([14]) then we obtain from (17)

$$\gamma \simeq \frac{1}{2}. \tag{18}$$

This value of γ is in good agreement with the corresponding value obtained in numerous experiments with moderate Reynolds number (see, for example, [18] and references therein). An analogous situation also takes place with the enstrophy field (cf. [14] and [18]).

It should be noted that another multifractal asymptotic can also be realized for the energy dissipation field ([14],[17]): $D_\infty^{(\varepsilon)} - d \simeq -1/2$. In this case one obtains from (9) that $\gamma \simeq 2/3$. One can also use the Meneveau relationship [16]

$$D_\infty^{(e)} = \frac{D_{2/3}^{(\varepsilon)} + 2D_\infty^{(\varepsilon)}}{3}. \tag{19}$$

Substituting (19) into (17) gives

$$\gamma = \frac{D_{2/3}^{(\varepsilon)} + 2D_{\infty}^{(\varepsilon)} - 3d}{3(D_{\infty}^{(\varepsilon)} - d)}. \tag{20}$$

Thus, multifractal asymptotics can be used to obtain PDF asymptotics and vice versa.

Clearly with increasing Reynolds numbers the character of dissipation field singularities may become increasingly sharp; that is, the dissipation field can 'lose' its multifractal asymptote with a definite value of $D_{\infty}^{(\varepsilon)}$. In this case, however, a smoother (positive) field

$$g(\mathbf{x}) = [ln\varepsilon(\mathbf{x})]^2$$

assumes the power-law singular (multifractal) form (10)-(11) with a definite value of $D_{\infty}^{([ln\varepsilon]^2)}$. Then, analogously to (16), we obtain (for large enough values of $[ln\varepsilon]^2$)

$$P(\varepsilon) \sim exp\{-b[ln\varepsilon]^{2\gamma}\} \tag{21}$$

where

$$\gamma = \frac{D_{\infty}^{(e)} - d}{D_{\infty}^{([ln\varepsilon]^2)} - d}. \tag{22}$$

(cf. (17)). As far as we know there is no information about the multifractality of the field $[ln\varepsilon]^2$. There is, however, information on the PDF of this field for large Reynolds number (see, for example, [1],[4],[5]). It follows from this information that the value of $\gamma \simeq 1$. Then (22) leads to

$$D_{\infty}^{(e)} \simeq D_{\infty}^{([ln\varepsilon]^2)}, \tag{23}$$

so for large Reynolds numbers the strongest singularities in the field $ln\varepsilon$ are the same as in the energy field.

5 Discussion

1. Analogous problems with log-normal-like probability density functions also appear for other extended systems with fractal (multifractal) behavior. These are (for example): the problem of large oceanic waves [19]; percolation in porous formations [20] and percolation

of random resistor networks (backbones) [21],[22]. The simple hierarchical models (such as the Novikov-Stewart model for turbulence [1] and the de Arcangelis et al. model for backbones of critical percolation clusters [21]) have analytically calculated multifractal asymptotics (i.e., definite D_∞ values) and, therefore, cannot be described in terms of log-normal-like PDFs (which cannot lead to definite D_∞). For the appearance of log-normal-like PDFs, strong (enough) singularities should appear in the field under consideration. For percolation networks (backbones) this phenomenon is related to their topology, and the cross-over to log-normal PDFs is related to topological changes in the networks.

2. On the other hand, one can expect that this cross-over could permit application of the Wyld ([23]) partial summation of the Feynman diagram series for corresponding fields (in our case it is the viscous dissipation field). The Dyson equation (in this approach) gives the famous Kraichnan asymptotic solution for the spectrum of turbulence *aaaray* [1]:

$$E_e \sim (u_0 \langle \varepsilon \rangle)^{1/2}\ k^{-3/2}$$

where $u_0 = \langle e \rangle^{1/2}$. This solution only approximately represents actual turbulent velocity fields (see [1] and Section 4), although the corresponding solution of the Dyson equation for the *aassaaataaa* field (after the cross-over)

$$E_\varepsilon \sim (u_0 \langle \varepsilon \rangle)^{3/2}\ k^{-1/2}$$

gives $\mu = 1/2$ (cf. (1)) which seems to be quite appropriate for turbulence with very large Reynolds numbers (see Introduction and Sections 4). The largest scale L (in this case, cf. (1)) is given by the Kolmogorov representation $L = u_0^3 / \langle \varepsilon \rangle$ [1].

This result allows one to hope that in other similar systems (oceanic waves, percolation networks, etc.) the Dyson equation can give relevant results after the cross-over as well.

3. Finally, let us briefly discuss the problem of passive scalar diffusion in the case when extreme values (hot spots, scalar particles) of the scalar field are concentrated along these caustics. In particular, we will consider the effective growth of the particle cloud along the caustic network. In [24] the relationship between a time of the diffusion growth,

t, and the characteristic size of this cloud, l, was obtained in the form

$$l \sim t^{\alpha}$$

where

$$\alpha = (D_{eff} - 1)^{-1}$$

in the three-dimensional space. To understand this relationship it is useful recall that for internal walks of a particle on the fractal

$$\langle r^2 \rangle \sim t^{2/D_w}$$

where r is the displacement of the particle in the space, D_w is the dimension of the *aataraaa* walks of the particle on the fractal. However, when we consider the effective growth of a *aaaua* of particles in three-dimensional space we should study the motion of some effective boundary surface of the cloud. In this case, the fractal dimension of the intersection of the effective (smooth: $D = 2$) boundary surface with the growing fractal plays the role of the fractal dimension of *axtaraaa*walks. Thus, we should replace D_w by $(D_{eff} - 1)$ when we replace r by l. Considering the effective diffusion coefficient of the passive scalar cloud [1]

$$K_{eff} = \frac{1}{6} \frac{dl^2(t)}{dt},$$

we obtain from these estimates

$$K_{eff} \sim l^{3-D_{eff}}.$$

Thus, for diffusion growth along the backbone ($D_{eff} = 5/3$, see Section 3)

$$K_{eff} \sim l^{4/3}.$$

This estimate coincides with the well known diffusion law of Richardson-Kolmogorov [1]. However, if one takes into account the dissipation rate singularities along the caustics then one obtains an increase in the effective diffusion coefficient K_{eff} with l (due to $D_{eff} < 5/3$ for the caustic networks with singularities, Section 3). The maximal acceleration of the effective diffusion takes place on the elastic backbone (defined by strongest stable singularities: pyramids and purses, Section 3). In this case $D_{eff} = 4/3$ and

$$K_{eff} \sim l^{5/3}.$$

It should be noted that such an enhanced effective diffusivity is characteristic of stratospheric observations (see, for example, [6] and references therein). When caustic networks degenerate the passive scalar clouds, naturally, cannot grow along them. However, a connection between the dissipation of passive scalar fluctuations and the viscous dissipation of kinetic energy still exists. Evidence of this connection is the log-normal PDF of the passive scalar dissipation and the value of exponent $\mu \simeq 0.5$ for scalar dissipation observed in turbulent mixing experiments [4] (cf. Section 4).

References

[1] A. C. Monin, and A. M. Yaglom, Statistical Fluid Mechanics, **2**, (MIT Press, Cambridge, 1975).

[2] K. R. Sreenivasan and P. Kailasnath, Phys. Fluids A, **5** (1993) 512.

[3] V. R. Kuznetsov, A. A. Praskovsky and V. A. Sabelnikov, J. Fluid Mech., **243** (1992) 595.

[4] C. H. Gibson, Proc. Roy. Soc. Lond. A, **434** (1991) 149.

[5] C. H. Gibson, Ocean Mixing and Kolmogorov's Intermittency Constant μ, Phys. Fluids (submitted).

[6] A. Bershadskii, E. Kit, and A. Tsinober, Proc. Roy. Soc. A, **441**, (1993) 147.

[7] V. I. Arnold, Singularities of Caustics and Wave Fronts, (Kluwer, 1990). See also much more simplified exposition in: V.I. Arnold, Catastrophe Theory (Springer-Verlag, 1986).

[8] D. Stauffer, and A. Aharony, Introduction to Percolation Theory, (Taylor and Francis, London 1992).

[9] A. G. Bershadskii, Sov. Phys. Usp., **33** (1990) 1073.

[10] A. J. Chorin, J. Stat. Phys., **69** (1992) 67.

[11] A. Bershadskii, Physica A, **206** (1994) 120.

[12] A. Bershadskii and H. Branover, J. de Physique I, 4 (1994) 1115.

[13] H. J. Herrmann, D. C. Hong and H. E. Stanley, J. Phys. A, **17** (1984) L261.

[14] A. Bershadskii et al., Fluid Dyn. Res., **14** (1994) 71.

[15] K. R. Sreenivasan, Annu. Rev. Fluid. Mech. **23** (1991) 539.

[16] C. Meneveau, J. Fluid Mech., **232** (1991) 469.

[17] A. Bershadskii and A. Tsinober, Phys. Rev. E, **48** (1993) 282.

[18] A. Bershadskii, E. Kit and A. Tsinober, Phys. Fluids A, **5** 1523 (1993).

[19] M.K. Ochi, Stochastic prediction of random seas, - In: Advances in Hydroscience, Urbana, V. 13 (1982) 217-375.

[20] G. Dagan, Flow and Transport in Porous Formations, (Springer-Verlag, 1989).

[21] L. de Arcangelis, S. Render and A. Coniglio, Phys. Rev. B, **31** (1985) 4725.

[22] R. Blumenfeld, J. Phys. A: Math. Gen., **21** (1988) 815.

[23] H. W. Wyld, Ann. Phys., **14** (1961) 143.

[24] B.M. Smirnov, Sov. Phys. Uspeckhi, **29** (1986) 481.